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Order parameter susceptibility in the Parisi solution for spin glasses

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Abstract. An exact expression is obtained for the order parameter susceptibility, χ_R, in the Parisi solution for spin glasses, which shows that the vanishing of the order parameter function, $q(x)$, at the origin, a constant spin susceptibility and the divergence of χ_R in the spin-glass phase are all intrinsically related. This expression for χ_R also suggests that the squared-averaged magnetisation is given by $q(0)$ and not $q(1)$.

1. Introduction

In the Parisi solution to the mean-field approximation for spin glasses (see Parisi 1981 for a review), the order parameter is a function, $q(x)$, $x \in [0, 1]$. The solution was first proposed for and applied to an infinite-range Ising spin-glass model for which the mean-field approximation is exact (Kirkpatrick and Sherrington 1978, hereafter referred to as SK), i.e. a model where each of the N Ising spins $S_i = \pm 1$, $i = 1, \ldots, N$, interacts with all the rest with quenched Gaussian random exchange interactions J_{ij} of zero mean and variance $1/N$. The thermodynamics of this quenched model are obtained by averaging the free energy over the Gaussian distribution of J_{ij} and this is accomplished by averaging the partition function for n identical replicas of the system ($\ln Z = \lim_{n\to 0}[(Z^n - 1)/n]$). The necessity for introducing an order parameter function arises because of instabilities arising in the limit $n \to 0$ (de Almeida and Thouless 1978, hereafter referred to as AT).

This unusual solution with an order parameter function has attracted a lot of attention and attempts have been made to understand its physical significance (Sompolinsky 1981). However, the behaviour of the order parameter susceptibility, χ_R (see below), in this solution has not been discussed sufficiently; the only calculation to date (Thouless *et al* 1980) was restricted to temperatures close to the spin-glass freezing temperature, T_g. However, χ_R is of great interest in the study of spin glasses: in simple solutions, where the order parameter function $q(x)$ is a constant independent of x (SK solution), χ_R diverges at T_g but goes negative for $T < T_g$ signalling the AT instability (Pytte and Rudnick 1979, Khurana and Hertz 1980); it is also related to the dynamic damping coefficient (Hertz *et al* 1982). An exact expression for χ_R in the Parisi solution is reported in this paper.

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By order parameter susceptibility, χ_R, in spin glasses, the trace of the square of the susceptibility matrix averaged over the distribution of exchange bonds is meant, i.e.

$$\chi_R=\sum_i \langle\chi_{ij}^2\rangle_{av} \qquad \chi_{ij}=\langle S_iS_j\rangle-\langle S_i\rangle\langle S_j\rangle. \tag{1.1}$$

(When two average signs appear together, as in $\langle\langle\ \ \rangle\rangle_{av}$, the inner stands for the thermal average, and the outer with a subscript av for the average over the quenched distribution of bonds. A single average sign (used later) would stand for averages done on an effective weight in replicated variables.) The expression for χ_R to be reported here relates it to the square of the second derivative of the free energy functional. The free energy per spin for the Parisi solution is written (Parisi 1980a)

$$-\beta F=\tfrac{1}{4}\beta^2\left(1+\int_0^1 q^2(x)\,dx-2q(1)\right)+C_{q(0)}(f(0,h)) \tag{1.2}$$

where $f(x,h)$ is a functional of $q(x)$ and satisfies a nonlinear differential equation

$$\frac{\partial f(x,h)}{\partial x}=-\tfrac{1}{2}\beta^2\frac{dq}{dx}\left[\frac{\partial^2 f(x,h)}{\partial h^2}+x\left(\frac{\partial f(x,h)}{\partial h}\right)^2\right] \tag{1.3}$$

with the boundary condition

$$f(1,h)=\ln(2\cosh h)$$

and $C_{q(0)}$ in (1.2) is a differential operator

$$C_q\equiv\exp(\tfrac{1}{2}\beta^2q\,\partial^2/\partial h^2).$$

Then

$$\chi_R^{-1}\propto 1-\beta^2C_{q(0)}(\partial^2f(0,h)/\partial h^2)^2. \tag{1.4}$$

Because $q(0)$ vanishes in zero field (§ 4 and Thouless *et al* 1980), the right-hand side of (1.4) will vanish in zero field (or χ_R will diverge) if

$$1=\beta^2(\partial^2f(0,h)/\partial h^2)^2. \tag{1.5}$$

It will be discussed later that for $q(0)=0$

$$\beta\frac{\partial^2f(0,h)}{\partial h^2}=\beta\left(1-\int_0^1 q(x)\,dx\right) \tag{1.6}$$

and since

$$\beta\left(1-\int_0^1 q(x)\,dx\right)\equiv\beta\langle\chi_{ii}\rangle_{av} \tag{1.7}$$

(Parisi 1980b), (1.5), (1.6) and (1.7) suggest that the condition for divergence of χ_R for all $T<T_g$ and a constant spin susceptibility independent of T for all $T<T_g$ are inseparable.

The close similarity of the expression on the right-hand side of (1.4) to the expression for χ_R in the SK solution (Khurana and Hertz 1980) suggests that the order parameter

$$q_{phy}\equiv\langle\langle s_i\rangle^2\rangle_{av}$$

is given by $q(0)$ contrary to the suggestion by Parisi, supported by Thouless *et al* (1980), identifying q_{phy} with $q(1)$. The name EA order parameter will not be used in this paper to avoid confusion with the dynamical connotations associated with the EA definition (Edwards and Anderson 1975).

It is hoped that this simple expression for χ_R will shed some new light on the nature of the spin-glass transition and the associated order parameter function; expansions near T_g, however useful, do not usually bring out relations that might exist between thermodynamic quantities. Thouless *et al* speculated on whether the divergence of χ_R would be suppressed by adding an external field $h(x)$ coupling to the order parameter function $q(x)$ by $\int_0^1 dx\, q(x)h(x)$. This speculation stems from the experience with Heisenberg ferromagnets where the susceptibilities transverse to the direction of magnetisation, m, diverge for $T < T_g$ in zero field, but acquire finite values m/h in the presence of a uniform field h. The expression (1.4) for χ_R suggests, however, that the divergence of χ_R depends crucially on $q(0)$ and not all of $q(x)$ and that χ_R has a physically acceptable non-negative value only for vanishing $q(0)$. As for the possibility of deriving the divergence of χ_R from the invariance of the functional $f(x, h)$ under some set of transformations, (1.4) says that such transformations might be nonlinear because χ_R depends nonlinearly on the derivatives of f. Alternatively, it might be possible to prove the constant value of the spin susceptibility invoking invariance under some linear transformations because it (the spin susceptibility) depends linearly on the derivatives of f and its constant value is a necessary condition for proving the divergence in χ_R.

The rest of the paper is divided into three further sections: in the first, Parisi's solution is briefly reviewed and some notation introduced, the expression (1.4) for χ_R is derived in the second, and the third section is devoted to some further discussion, especially to the identification of q_{phy}.

2. Parisi's solution

Parisi's solution starts from the expression for the free energy per spin for the SK model in the presence of a uniform field h (Kirkpatrick and Sherrington 1978)

$$-\beta F = \tfrac{1}{4}\beta^2 - \lim_{n\to 0}\frac{1}{n}\max\left[\sum_{(\alpha\beta)} \tfrac{1}{2}q_{\alpha\beta}^2 = \ln \operatorname{Tr} \exp\left(\beta \sum_{(\alpha\beta)} S_\alpha S_\beta q_{\alpha\beta} + \sum_\alpha hS_\alpha\right)\right], \tag{2.1}$$

$\alpha, \beta = 1, \ldots, n$ and the sums are over distinct pairs of α and β, and proceeds by choosing the saddle-point values of $q_{\alpha\beta}$ to depend on α, β in a definite way. For convenience in keeping track of replica indices in calculating χ_R, the scheme outlined below differs slightly from the one proposed by Parisi; however, the difference is of no consequence. In the following, the $n \times n$ matrix $q_{\alpha\beta}$ at the saddle point will be divided into $n_1 \times n_1$ blocks, p_1^2 in number, so that $p_1 n_1 = n$, the diagonal elements of the off-diagonal blocks will be called q_0, and the rest of the elements q_1. Each of the p_1^2 blocks will be further divided into $n_2 \times n_2$ blocks, p_2^2 in number, so that $p_2 n_2 = n_1$, the diagonal elements of the new off-diagonal blocks will continue to have the value q_1, but all off-diagonal elements of each of the $(p_1 p_2)^2$ blocks will have the value q_2. This procedure will be iterated; each step in the series of iterations will be labelled by an integer K and described by a function $G_K(h, h_{\alpha_K}^{a_K})$ calculated in the presence of a field $h_{\alpha_K}^{a_K}$ which couples to the replicas (labelled by α_K) in the smallest blocks (labelled by a_K) at any stage of iteration; and the contribution of the ln Tr term in (2.1) will be

$$\lim_{n\to 0} (1/n) \ln G_K(h) - \tfrac{1}{2}\beta^2 q_0. \tag{2.2}$$

For $K=1,2$

$$G_1(h, h^{a_1}_{\alpha_1}) = C_{q_1} \prod_{\alpha_1=1}^{n_1} \left(C_{q_0-q_1} \prod_{a_1=1}^{p_1} [2\cosh(h + h^{a_1}_{\alpha_1})] \right) \tag{2.3}$$

$$G_2(h, h^{a_2}_{\alpha_2}) = C_{q_2} \prod_{\alpha_2=1}^{n_2} \left[C_{q_1-q_2} \prod_{a_2=1}^{p_2} \left(C_{q_0-q_1} \prod_{a_1=1}^{p_1} [2\cosh(h + h^{a_2}_{\alpha_2})] \right) \right]. \tag{2.4}$$

When the replica-dependent fields $h^{a_K}_{\alpha_K}$ are set equal to zero, $G_K(h)$ (fields will be assumed to have been set equal to zero when not explicitly written in the arguments of the functions) reduce to similar functions in Parisi's solution (Duplantier 1981), i.e.

$$G_K(h) = C_{q_K}(g_K(h)) \tag{2.5}$$

$$g_K(h) = C_{q_{K-1}-q_K}(g_{K-1}(h))^{m_K/m_{K-1}} \tag{2.6}$$

if the m_K variables in Parisi's solution are defined by

$$m_K = \prod_{i=1}^{K} p_i \qquad \begin{cases} m_0 = 1 \\ m_{K+1} = 0. \end{cases} \tag{2.7}$$

Here Parisi's condition

$$m_1 \geqslant m_2 \ldots \geqslant m_K$$

is automatically satisfied if $p_i \leqslant 1$.

The replica-dependent fields, $h^{a_K}_{\alpha_K}$, have been introduced to facilitate taking averages over the effective weight in replica variables, S_α, given by the ln Tr term in (2.1). Indeed, it is trivial to check that equations of state for order parameters, q_i, obtained by taking double derivatives of $G_K(h^{a_K}_{\alpha_K}, h)$ with respect to $h^{a_i}_{\alpha}$, $i = 1, \ldots, K$, setting $h^{a_i}_{\alpha_i} = 0$ and taking the limit $n \to 0$ are the same as obtained by substituting (2.2) in (2.1) and varying the free energy with respect to q_i. This procedure for calculating averages in replicated variables S_α will be used in the calculation of χ_R. This section will now be closed with an additional comment that at any stage of iteration K, the order parameter q_K is given by

$$q_K = C_{q_K}[(1/m_K)(\partial/\partial h)\ln g_K(h)]^2 \tag{2.8}$$

with $g_K(h)$ independent of q_K, and that

$$1 - \sum_{i=0}^{K} (m_i - m_{i+1}) q_i = C_{q_K}[(1/m_K)(\partial^2/\partial h^2)\ln g_K(h)]. \tag{2.9}$$

It may also be recalled (Duplantier 1981) that the series of iterations (2.3)–(2.4) for $h^{a_K}_{\alpha_K} = 0$ when substituted in (2.1) reproduce (1.3) in the limit $K \to \infty$ with

$$f(x, h) = (1/x)\ln g(x, h). \tag{2.10}$$

Then (2.8) and (2.9) give

$$q(0) = C_{q(0)}(\partial f(0, h)/\partial h)^2 \tag{2.11}$$

$$1 - \int_0^1 q(x)\,\mathrm{d}x = C_{q(0)}(\partial^2 f(0, h)/\partial h^2). \tag{2.12}$$

(2.12) reduces to (1.6) when $q(0) = 0$.

3. Order parameter susceptibility

One problem in calculating χ_R as defined by (1.1) in solutions which break the permutation symmetry between different replicas has been to identify the correlation in replica variables, $S^{a}_{\alpha_i}$, that corresponds to χ_R, especially since (2.1) is independent of site variables i and j. The following two points are useful in arriving at a representation for χ_R.

(i) In the SK solution, the expression for χ_R reads (Khurana and Hertz 1980)

$$\chi_R = \Pi(q)/(1-\beta^2\Pi(q)) \tag{3.1}$$

where

$$\Pi(q) = \langle \chi_{ii}^2 \rangle_{\mathrm{av}}. \tag{3.2}$$

This decomposition of χ_R into a single-site spin correlation depends on the infinite range of the SK model and is not spoilt by any manipulations on replica indices. Hence, only knowledge of $\Pi(q)$ is required in calculating χ_R.

(ii) As discussed by Parisi (1980b) and mentioned earlier in (1.7), the single-site spin–spin correlation in this solution reads

$$\langle \chi_{ii} \rangle_{\mathrm{av}} = \lim_{n\to 0} \frac{1}{n} \sum_{\alpha,\beta=1}^{n} \langle S_\alpha S_\beta \rangle \qquad S_\alpha^2 = 1. \tag{3.3}$$

The average on the right-hand side is taken with weights given by the Tr term in (2.1). Since the usual way the average (over a quenched distribution) of the square of a thermally averaged variable is obtained in the replica formalism is by introducing two different groups of replicas (Kirkpatrick and Sherrington 1978), $\Pi(q)$ in replica formalism should read

$$\Pi(q) = \lim_{n\to 0} \frac{1}{n^2} \sum_{\alpha,\beta=1}^{n} \sum_{\gamma,\delta=1}^{n} \langle S_\alpha S_\beta S_\gamma S_\delta \rangle. \tag{3.4}$$

(α, β) and (γ, δ) belong to two different groups of replicas, each group running $1, \ldots, n$ but no value of (α, β) equals any value of (γ, δ).

The expression (3.4) for $\Pi(q)$ has a sum on replica indices running from $1, \ldots, n$ and in any solution which breaks the permutation symmetry between replicas, this sum may be expanded into a variety of terms corresponding to correlations of spins within a block and those between spins in different blocks. One may also define several partial correlations, for example, when spins within only one block are considered. However, calculations like the one outlined here for $\Pi(q)$ suggest that all such partial correlations are 'benign' in that they have stable values (less than $1/\beta^2$) even when $\Pi(q) > 1/\beta^2$ and χ_R is unstable.

Before going to the details of the calculation for $\Pi(q)$, it should be pointed out that this calculation, by itself, neither identifies these partial correlations with any physical quantity, nor says whether they arise from some kind of 'restricted' statistical mechanical averaging. (The latter possibility arises because computer simulations show the phase space for a spin glass to consist of a large number of energy minima separated by barriers which are infinitely high in the thermodynamic limit (Mackenzie and Young 1982).) Similarly, this calculation by itself does not reveal the nature of the statistical mechanical averaging defined by equations (3.3) and (3.4). However, comparing the results of this calculation with those of dynamical calculations suggests that $\Pi(q)$ defined in equation (3.4) corresponds to the dynamical damping coefficient at the (longest) infinite time scales whereas the partial correlations correspond to damping coefficients which do not truly probe this infinite time scale (Khurana 1983, Sommers 1982). This correspondence seems to imply that (3.3) and (3.4) provide a way for doing statistical mechanical averages over the set of states to which a spin glass relaxes in this infinite time limit. Relations among various approaches that have been used to study spin glasses should be explored further for any evidence substantiating (or refuting) this suggestion.

The scheme outlined in § 2 for breaking the $n\times n$ matrices in (α,β) and (γ,δ) may now be applied to (3.4) and $\Pi(q)$ calculated. Writing out explicitly all possible combinations of a pair of replicas yields for $K=1$

$$\sum_{\substack{\alpha,\beta\\ \gamma,\delta}}\langle S_\alpha S_\beta S_\gamma S_\delta\rangle$$

$$\begin{aligned}
&=n^2+2n\sum_{a_1\neq b_1;\alpha_1}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\alpha_1}\rangle+2n\sum_{\alpha_1\neq\beta_1}\Big(\sum_{a_1}\langle S^{a_1}_{\alpha_1}S^{a_1}_{\beta_1}\rangle+\sum_{a_1\neq b_1}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\beta_1}\rangle\Big)\\
&+2\sum_{\substack{\alpha_1\neq\beta_1\\ \gamma_1}}\Big(\sum_{\substack{a_1\\ c_1\neq d_1}}\langle S^{a_1}_{\alpha_1}S^{a_1}_{\beta_1}S^{c_1}_{\gamma_1}S^{d_1}_{\gamma_1}\rangle+\sum_{\substack{a_1\neq b_1\\ c_1\neq d_1}}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\beta_1}S^{c_1}_{\gamma_1}S^{d_1}_{\gamma_1}\rangle\Big)\\
&+\sum_{\substack{a_1\neq b_1;\alpha_1\\ c_1\neq\alpha_1;\gamma_1}}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\beta_1}S^{c_1}_{\gamma_1}S^{d_1}_{\gamma_1}\rangle+\sum_{\substack{\alpha_1\neq\beta_1\\ \gamma_1\neq\delta_1}}\Big(\sum_{\substack{a_1\\ c_1}}\langle S^{a_1}_{\alpha_1}S^{a_1}_{\beta_1}S^{c_1}_{\gamma_1}S^{c_1}_{d_1}\rangle\\
&+2\sum_{\substack{a_1\neq b_1\\ c_1}}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\beta_1}S^{c_1}_{\gamma_1}S^{c_1}_{\delta_1}\rangle+\sum_{\substack{a_1\neq b_1\\ c_1\neq d_1}}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\beta_1}S^{c_1}_{\gamma_1}S^{d_1}_{\delta_1}\rangle\Big).
\end{aligned}\tag{3.5}$$

(a_1,b_1) label blocks in the set of replicas (α,β) and take values from 1 to p_1; (α_1,β_1) label indices within each of the p_1^2 blocks and take values from 1 to n_1. (c_1,d_1), (γ_1,δ_1) do the same for the group (γ,δ). Factors of 2 in (3.5) arise from the symmetry between the two groups. Each term in (3.5) can be evaluated by differentiating $G_1(h,h^{a_1}_{\alpha_1})$. The terms under summation signs are degenerate. For example, all terms of the type $\langle S^{a_1}_{\alpha_1}S^{b_1}_{\alpha_1}S^{c_1}_{\gamma_1}S^{d_1}_{\gamma_1}\rangle$, $a_1\neq b_1$, $c_1\neq d_1$, are the same as $\langle S^1_1S^2_1S^3_2S^4_2\rangle$ when $h^{a_1}_{\alpha_1}=0$. Now

$$\begin{aligned}
\langle S^1_1S^2_1S^3_2S^4_2\rangle&=\lim_{n\to0}\frac{\partial}{\partial h^1_1}\frac{\partial}{\partial h^2_1}\frac{\partial}{\partial h^3_2}\frac{\partial}{\partial h^4_2}G(h,h^{a_1}_{\alpha_1})|_{h^{a_1}_{\alpha_1}=0}\\
&=C_{q_1}\Big(\frac{1}{g_1(h)}C_{q_0-q_1}[\tanh^2h(2\cosh h)^{p_1}]\Big)\\
&=C_{q_1}\Big[\frac{1}{p_1(p_1-1)}\Big(\frac{1}{g_1(h)}\frac{\partial^2g_1(h)}{\partial h^2}-p_1\Big)\Big].
\end{aligned}$$

Similarly all terms of the type $\langle S^{a_1}_{\alpha_1}S^{b_1}_{\beta_1}S^{c_1}_{\gamma_1}D^{d_1}_{\delta_1}\rangle$, $a_1\neq b_1$, $c_1\neq d_1$, $\alpha_1\neq\beta_1$, $\gamma_1\neq\delta_1$, are degenerate with $\langle S^1_1S^2_2S^3_3S^4_4\rangle$ given by

$$\begin{aligned}
\langle S^1_1S^2_2S^3_3S^4_4\rangle&=\lim_{n\to0}\frac{\partial}{\partial h^1_1}\frac{\partial}{\partial h^2_2}\frac{\partial}{\partial h^3_3}\frac{\partial}{\partial h^4_4}G(h,h^{a_1}_{\alpha_1})|_{h^{a_1}_{\alpha_1}=0}\\
&=C_{q_1}\Big(\frac{C_{q_0-q_1}[(2\cosh h)^{p_1}\tanh h]}{g_1(h)}\Big)^2\\
&=C_{q_1}\Big[\frac{1}{p_1}\frac{\partial}{\partial h}\ln g_1(h)\Big]^2.
\end{aligned}$$

Following this procedure for each term in (3.5), $\Pi(q)$ for $K=1$ reads

$$\Pi_1(q_0, q_1) = C_{q_1}\Big\{1 + 2(p_1-1)\frac{1}{p_1(p_1-1)}\Big(\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2} - p_1\Big) - 2p_1\Big(\frac{1}{p_1}\frac{\partial}{\partial h}\ln g_1(h)\Big)^2$$
$$-2\Big(\frac{1}{p_1}\frac{\partial}{\partial h}\ln g_1(h)\Big)^2\Big(\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2} - p_1\Big)$$
$$+(p_1-1)^2\Big[\frac{1}{p_1(p_1-1)}\Big(\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2} - p_1\Big)\Big]^2 + \Big(\frac{1}{p_1}\frac{\partial \ln g_1(h)}{\partial h}\Big)^4$$
$$+(p_1-1)^2\Big(\frac{1}{p_1}\frac{\partial}{\partial h}\ln g_1(h)\Big)^4 + 2(p_1-1)\Big(\frac{1}{p_1}\frac{\partial \ln g_1(h)}{\partial h}\Big)^4\Big\}$$
$$= C_{q_1}\Big\{\Big[1 + \frac{1}{p_1}\Big(\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2} - p_1\Big)\Big]^2 + \frac{1}{p_1^2}\Big(\frac{\partial \ln g_1(h)}{\partial h}\Big)^4$$
$$-2\frac{1}{p_1^2}\Big(\frac{\partial \ln g_1(h)}{\partial h}\Big)^2\Big(\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2}\Big)\Big\}$$
$$= C_{q_1}\Big\{\frac{1}{m_1}\Big[\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2} - \Big(\frac{\partial}{\partial h}\ln g_1(h)\Big)^2\Big]\Big\}^2$$
$$= C_{q_1}\Big(\frac{1}{m_1}\frac{\partial^2}{\partial h^2}\ln g_1(h)\Big)^2. \tag{3.6}$$

This symmetry in different derivatives of $g_1(h)$ which led to this simple and compact expression for $\Pi_1(q)$ holds at every step in the iterations leading to Parisi's solution. As further evidence for this statement, the terms for $K=2$ are displayed below:

$$\Pi_2(q_0, q_1, q_2) = \lim_{n\to 0}\frac{1}{n^2}\Big\{\Big[n^2 + 2n\Big(\sum_{a_1\neq b_1;\alpha_1}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\alpha_1}\rangle + \sum_{a_1,b_1}\sum_{a_2\neq b_2;\alpha_2}\langle S^{a_2}_{\alpha_2}S^{b_2}_{\alpha_2}\rangle\Big)$$
$$+\Big(\sum_{\substack{a_1\neq b_1;\alpha_1\\ c_1\neq d_1;\gamma_1}}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\alpha_1}S^{c_1}_{\gamma_1}S^{d_1}_{\gamma_1}\rangle + \sum_{\substack{a_1,b_1\\ c_1,d_1}}\sum_{\substack{a_2\neq b_2;\alpha_2\\ c_2\neq d_2;\gamma_2}}\langle S^{a_2}_{\alpha_2}S^{b_2}_{\alpha_2}S^{c_2}_{\gamma_2}S^{d_2}_{\gamma_2}\rangle$$
$$+2\sum_{\substack{a_1\neq b_1;\alpha_1\\ c_1,d_1}}\sum_{c_2\neq d_2;\gamma_2}\langle S^{a_1}_{\alpha_1}S^{b_1}_{\alpha_1}S^{c_2}_{\gamma_2}S^{d_2}_{\gamma_2}\rangle\Big)\Big]$$
$$+2\sum_{\substack{a_1,b_1\\ a_2,b_2;\alpha_2\neq\beta_2}}\Big(\langle S^{a_2}_{\alpha_2}S^{b_2}_{\beta_2}\rangle + \sum_{\substack{c_1,d_1\\ c_2\neq d_2;\gamma_2}}\langle S^{a_2}_{\alpha_2}S^{b_2}_{\beta_2}S^{c_2}_{\gamma_2}S^{d_2}_{\gamma_2}\rangle$$
$$+\sum_{c_1\neq d_1;\gamma_1}\langle S^{a_2}_{\alpha_2}S^{b_2}_{\beta_2}S^{c_1}_{\gamma_1}S^{d_1}_{\gamma_1}\rangle\Big) + \sum_{\substack{a_1,b_1\\ c_1,d_1}}\sum_{\substack{a_2,b_2;\alpha_2\neq\beta_2\\ c_2,d_2;\gamma_2\neq\delta_2}}\langle S^{a_2}_{\alpha_2}S^{b_2}_{\beta_2}S^{c_2}_{\gamma_2}S^{d_2}_{\delta_2}\rangle\Big\}$$
$$= C_{q_2}\Big[\Big[\Big\{1 + \frac{1}{m_1 g_2(h)}C_{q_1-q_2}\Big[(g_1(h))^{m_2/m_1}$$
$$\times\Big(\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2} - m_1 + \frac{m_2-m_1}{m_1}\Big(\frac{\partial \ln g_1(h)}{\partial h}\Big)^2\Big)\Big]\Big\}^2$$
$$-\frac{2}{m_2}\Big(\frac{\partial \ln g_2(h)}{\partial h}\Big)^2\Big\{1 + \frac{1}{m_1 g_2(h)}C_{q_1-q_2}\Big[(g_1(h))^{m_2/m_1}\Big(\frac{1}{g_1(h)}\frac{\partial^2 g_1(h)}{\partial h^2} - m_1$$

$$+\frac{m_2-m_1}{m_1}\left(\frac{\partial \ln g_1(h)}{\partial h}\right)^2\Bigg)\Bigg]\Bigg\}+\frac{1}{m_2^2}\left(\frac{\partial \ln g_2(h)}{\partial h}\right)^4\Bigg].$$

$$=C_{q_2}\left\{\frac{1}{m_2^2}\left[\frac{1}{g_2(h)}\frac{\partial^2 g_2(h)}{\partial h^2}-\left(\frac{\partial \ln g_2(h)}{\partial h}\right)^2\right]^2\right\}$$

$$=C_{q_2}\left(\frac{1}{m_2}\frac{\partial^2 \ln g_2(h)}{\partial h^2}\right)^2. \tag{3.7}$$

It should be obvious from (3.6) and (3.7) that after any finite number of iterations K, $\Pi_K(q)$ is given by

$$\Pi_K(q)=C_{q_K}\left(\frac{1}{m_K}\frac{\partial^2}{\partial h^2}\ln g_K(h)\right)^2. \tag{3.8}$$

In the limit $K\to\infty$, one may define an x-dependent function

$$\Pi(x,q)=C_{q(x)}\left(\frac{\partial^2}{\partial h^2}f(x,h)\right)^2 \qquad f(x,h)=(1/x)\ln g(x,h) \tag{3.9}$$

which reproduces the result for the replica symmetric SK solution for $x=1$ and should be the answer for the Parisi solution for $x=0$.

4. Order parameters and stability

If a stable, physically acceptable solution must have a non-negative value for χ_R, this condition for stability now reads

$$1\geqslant\beta^2\Pi_K(q). \tag{4.1}$$

The expression (3.8) for $\Pi_K(q)$ after a finite number K of iterations together with the equation of state (2.8) for q_K shows that such a solution is unstable just like the SK solution was unstable and for the same reason, namely C_{q_K} giving different combinatorial weights to terms of order $(q_K)^n$ in (3.8) than to those of order $(q_K)^{n+1}$ in (2.8), when the right-hand sides of (3.8) and (2.8), respectively, are expanded in powers of q_K. This discrepancy was singled out by Khurana and Hertz (1980) as the source of the AT instability, and this analysis shows that it is not removed until $q(x)$ vanishes, i.e. until Parisi's replica symmetry breaking ansatz has been iterated infinitely many times. This analysis also shows that (3.10) would be violated and the solution would be unstable if $q(0)$ were non-zero in zero field, unless $(\partial^4 f(0,h)/\partial h^4)|_{h=0}$ were zero. However, analysis of the equation of state (2.11) for $q(0)$ shows that $q(0)=0$ is a solution of this equation, even for $T<T_g$, if

$$1=\beta^2(\partial^2 f(0,h)/\partial h^2)^2|_{h=0}. \tag{4.2}$$

This may be understood by comparing (2.11) with the equation of state for the SK order parameter. In this formalism, that equation reads

$$q_{SK}=C_{q_{SK}}(\partial f(1,h)/\partial h)^2 \tag{4.3}$$

($f(1,h)=\ln(2\cosh h)$ was defined after (1.3)). Expanding the right-hand side of (4.3) for small q_{SK} gives the well known result

$$q_{SK} \propto |1 - \beta^2(\partial^2 f(1, h)/\partial h^2)^2_{h=0}|$$

for

$$1 < \beta^2(\partial^2 f(1, h)/\partial h^2)^2_{h=0}.$$

This also defines the spin-glass freezing temperature T_g; $T_g = 1$ because $(\partial^2 f(1, h)/\partial h^2)|_{h=0} = 1$. A similar expansion for (2.11) would then give

$$q(0) = 0$$

if (4.2) were satisfied. That $q(0)$ vanishes in zero field is also obtained when examining the extremum of the free energy functional (1.2) close to T_g (Parisi 1980b, Thouless *et al* 1980).

Thus the vanishing of the order parameter function, $q(x)$, at the origin, a constant temperature-independent spin susceptibility, $\langle\chi_{ii}\rangle_{av}$, for $T < T_g$ and a divergent order parameter susceptibility, χ_R, for $T < T_g$ are all intrinsically related. These are, in a way, different expressions of the same statement. This is a feature which approximations close to T_g do not bring out. This close connection makes one wonder whether the condition for overcoming the problem of negative zero temperature entropy in the SK solution (Kirkpatrick and Sherrington 1978) is also just a part in this same connection.

One may ask what one learns about the order parameter function from this discussion of the order parameter susceptibility. Setting $x = 1$ in (3.9) reproduces the expression for $\Pi(q)$ in the SK solution if $q(1)$ is identified with the order parameter appearing in the SK solution. Since SK argued that the order parameter in their solution was equal to $\langle\langle S_i\rangle^2\rangle_{av}$, one may be tempted to conclude that the above analysis justifies Parisi's identification of $q(1)$ with $\langle\langle S_i\rangle^2\rangle_{av}$. However, the following argument based on the analysis of χ_R and $\Pi(q)$ in the SK solution without the use of replicas (Khurana and Hertz 1980) shows that this identification is not self-consistent.

At high temperatures, Π is just the product of two two-point vertices in the Ising model. When $T < T_g$ and the expectation value $\langle\langle S_i\rangle\rangle$ of a local spin is different from zero, these condensate lines are attached to the two-point vertices and connecting these in pairs gives factors of $\langle\langle S_i\rangle^2\rangle_{av}$. That (1.4) has the same form as that obtained for the SK solution without the use of replicas then suggests that these arguments are also valid in the Parisi solution. The difference from the SK solution is that the two-point vertex is $\partial^2 f(0, h)/\partial h^2$ and that a pair of condensate lines gives $q(0)$. Thus in the limit of a stable solution, $\langle\langle S_i\rangle^2\rangle_{av}$ is given by $q(0)$.

The connection between χ_R and the damping coefficient in dynamic theories will be taken up separately. One would expect that the relation between χ_R and the damping coefficient found in the SK solution (Hertz *et al* 1982) also holds for solutions with replica symmetry breaking, but this calculation of χ_R does not agree with the calculation of an x-dependent damping coefficient (Hertz 1983) which diverges for all x between 0 and 1.

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